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Some Properties of General Convolution in Spatial and Frequency Domains Associated with Hypercomplex Fourier Transform

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Abstract

The correspondence between quaternion convolution and quaternion product associated with the hypercomplex Fourier transforms is studied. Some useful properties of relationship between quaternion convolution and the hypercomplex Fourier transform are obtained.

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1 Introduction

It is well-known that the hypercomplex Fourier transform or the quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using quaternion algebra. The QFT has been shown to relate to the other quaternion signal analysis tools such as quaternion wavelet transform, fractional quaternion Fourier transform, quaternionic windowed Fourier transform and quaternion Wigner-Ville distribution [8, 9, 10]. According to the non-commutative property of quaternion multiplication, there are at least three different types of two-dimensional QFT as follows (see, for example, [6])

$$\mathcal{F}_q^{I}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu_1\omega_1 x_1}\, f(\boldsymbol{x})\, e^{-\mu_2\omega_2 x_2}\, d\boldsymbol{x}, \quad d\boldsymbol{x} = dx_1 dx_2, \tag{1}$$

$$\mathcal{F}_q^{II}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-\mu_1 \boldsymbol{\omega}\cdot\boldsymbol{x}}, \quad \boldsymbol{\omega}\cdot\boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2, \tag{2}$$

$$\mathcal{F}_q^{III}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mu_1 \boldsymbol{\omega}\cdot\boldsymbol{x}}\, f(\boldsymbol{x})\, d\boldsymbol{x}, \quad \boldsymbol{\omega}\cdot\boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2, \tag{3}$$

where μ_1 and μ_2 are any two unit pure quaternions ($\mu_1^2 = \mu_2^2 = -1$) that are orthogonal to each other. These three QFTs are type I, II, and III, respectively.

In [5], authors have established convolution theorems for the type I QFT, which describes how to connect the convolution definition in the quaternion spatial and quaternion frequency domains. Another approach of the convolution theorem for the type I QFT have been recently proposed in [2]. They use the decompositions of commuting and anticommuting of quaternions. It is shown that these theorems are not only valid for real-valued functions, but also for full quaternion-valued functions. In [11], the authors derived correlation theorem using quaternion convolution associated with the type II QFT. General convolution for the Clifford Fourier transform (CFT) was already proposed in [3, 4]. However, some important properties of general convolution of connecting quaternion spatial and quaternion frequency, similar to the version of the classical convolution, are still to be established. In the present paper we begin by establishing convolution theorem in the hypercomplex Fourier domains. Using the decomposition of quaternion functions and their quaternion Fourier transform we derive some useful properties of convolution theorem associated with the type II QFT. Because quaternion multiplication is not commutative, we find that the properties are significantly different from the classical case [1].

2 Quaternion

2.1 Basic Properties of Quaternion

The quaternion, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843. It is a generalization of complex number to a 4D algebra and is denoted by $\mathbb{H}$. Every element of $\mathbb{H}$ can be written in a hypercomplex form as follows

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 : \ q_0, q_1, q_2, q_3 \in \mathbb{R}\}. \tag{4}$$

Here the three different imaginary parts satisfy the following multiplication rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j},\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{5}$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar* part of q denoted by $Sc(q)$ and a *pure quaternion* $\mathbf{q}$ denoted by $\mathrm{Vec}(q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$.

The *quaternion conjugate* of a quaternion q is given by

$$\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3. \tag{6}$$

Any $q \in \mathbb{H}$ can be represented in polar form as

$$q = |q|e^{\mu\theta}, \quad |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}, \tag{7}$$

where

$$\cos\theta = \frac{q_0}{|q|}, \quad \sin\theta = \frac{\sqrt{q_1^2 + q_2^2 + q_3^2}}{|q|}. \tag{8}$$

Then, μ can be expressed as

$$\mu = \frac{q_0}{\sqrt{q_1^2 + q_2^2 + q_3^2}} + \mathbf{i}\frac{q_1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} + \mathbf{j}\frac{q_2}{\sqrt{q_1^2 + q_2^2 + q_3^2}} + \mathbf{k}\frac{q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}. \tag{9}$$

Notice that μ is a unit pure quaternion and it is referred to the eigen-axis which represents the direction in the 3-D space of imaginary part of quaternion number. Notice also that θ is referred to the eigen-angle.

2.2 Commuting and anticommuting of quaternion and properties

Let us briefly discuss some of the basic formulas of commuting and anticommuting of quaternion, which will be used to prove the main results. For clarity, we introduce the following definition.

Definition 2.1. *Let $q \in \mathbb{H}$ and an arbitrary square root μ be an arbitrary square root, namely $\mu^2 = -1$. We define*

$$q_{c^l\mu} = \frac{1}{2}(q - (-1)^l \mu q \mu). \tag{10}$$

Based on (10), any $q \in \mathbb{H}$ can be split up into the commuting ($l = 0$) and anticommuting ($l = 1$) parts (see [2, 4, 7]) with respect with μ as follows:

$$\begin{aligned} q &= q_{c^0\mu} + q_{c^1(\mu)} \\ &= q_- + q_+. \end{aligned} \tag{11}$$

Here, we introduced the notations $q_{c^0(\mu)} = q_-$ and $q_{c^1\mu} = q_+$. We also have

$$\begin{aligned} \mu q_{c^0\mu} &= \mu q_- = q_- \mu, \\ \mu q_{c^1\mu} &= \mu q_+ = -q_+ \mu. \end{aligned} \tag{12}$$

The first part of (12) can be easily seen by

$$\begin{aligned} \mu q_{c^0\mu} &= \frac{1}{2}\mu(q - \mu q \mu) \\ &= \frac{1}{2}(\mu q - \mu^2 q \mu) \\ &= \frac{1}{2}(q\mu + \mu q) \\ &= \frac{1}{2}(q - \mu q \mu)\mu \\ &= q_{c^0\mu}\mu. \end{aligned} \tag{13}$$

The commuting and anticommuting parts satisfy the interesting properties:

$$\mu^2 = \mu_-^2 + \mu_+^2 = -1, \qquad \mu_+\mu_- + \mu_-\mu_+ = 0. \tag{14}$$

Let us give an alternative proof of the above identity (compare to [2]). For a square root ν of -1, namely $\nu^2 = -1$, we have $\mu_{c^0\nu} = \mu_-$ and $\mu_{c^1\nu} = \mu_+$. It implies that

$$\begin{aligned} \mu_-^2 &= \frac{1}{4}(\mu - \nu\mu\nu)^2 \\ &= \frac{1}{4}(\mu^2 - \mu\nu\mu\nu - \nu\mu\nu\mu - \nu\mu\nu\nu\mu\nu) \\ &= \frac{1}{4}(\mu^2 + \mu\nu^2\mu + \nu\mu^2\nu - 1) \\ &= \frac{1}{4}(\mu^2 + 1) \\ &= 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mu_+^2 &= \frac{1}{4}(\mu + \nu\mu\nu)^2 \\ &= \frac{1}{4}(\mu^2 - \mu\nu^2\mu - \nu\mu^2\nu - 1) \\ &= \frac{1}{4}(\mu^2 - 1 - 1 - 1) \\ &= -1. \end{aligned} \tag{16}$$

For the special case $\mu_{c^0\mu} = \mu_-$ and $\mu_{c^1\mu} = \mu_+$, we have $\mu_-^2 = -1$ and $\mu_+^2 = 0$. Hence

$$\mu_-^2 + \mu_+^2 = -1. \tag{17}$$

Observe that

$$\mu^2 = (\mu_- + \mu_+)^2 = \mu_-^2 + \mu_+\mu_- + \mu_-\mu_+ + \mu_+^2. \tag{18}$$

Then, inserting (17) into (18), we have (14). Therefore,

$$q_\pm e^{\mu\theta} = e^{\mp\mu\theta} q_\pm. \tag{19}$$

For a multi-index $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_1 \in \{0,1\}$, define a function $f^\varphi : \mathbb{R}^2 \to \mathbb{H}$ by

$$f^\varphi = f^{(\varphi_1,\varphi_2)} = f((-1)^{\varphi_1}x_1, (-1)^{\varphi_2}x_2). \tag{20}$$

3 Main Results

Let us introduce the quaternion Fourier transform (QFT) and the quaternion convolution (compare to [5, 11]).

Definition 3.1. *The type II QFT of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is the transform $\mathcal{F}_q\{f\} : \mathbb{R}^2 \longrightarrow \mathbb{H}$ given by the integral*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x}, \tag{21}$$

where μ is any unit pure quaternion such that $\mu^2 = -1$. The inverse transform of the QFT is given by

$$\begin{aligned} f(\boldsymbol{x}) &= \mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](\boldsymbol{x}) \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}. \end{aligned} \tag{22}$$

Definition 3.2 (Quaternion Convolution)**.** *Let f, $g \in L^1(\mathbb{R}^2;\mathbb{H})$. The convolution $f \star g$ of f and g is defined by*

$$(f \star g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y}. \tag{23}$$

Next, we derive the convolution theorem for the QFT. Some relationships between the quaternion convolution and its QFT are also presented.

Theorem 3.1. *Let f, $g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then we have*

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}) \\ &\quad + \mathbf{j}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega}). \end{aligned} \tag{24}$$

Proof. Denote by $\mathcal{F}_q\{f\}$ and $\mathcal{F}_q\{g\}$, the type II QFT of f and g, respectively. Expanding the type II QFT of the left-hand sided of (24), we obtain

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &\overset{(21)}{=} \int_{\mathbb{R}^2} (f \star g)(\boldsymbol{x})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}\right] e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{x} - \boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x}\right] d\boldsymbol{y}. \end{aligned} \tag{25}$$

Changing variables $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ in the above expression, we get

$$\begin{aligned}
\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{z})\, e^{-\mu(\boldsymbol{\omega}\cdot\boldsymbol{y}+\boldsymbol{\omega}\cdot\boldsymbol{z})}\, d\boldsymbol{z} \right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{z} \right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} [f_0(\boldsymbol{y}) + \mathbf{i}f_1(\boldsymbol{y}) + \mathbf{j}f_2(\boldsymbol{y}) + \mathbf{k}f_3(\boldsymbol{y})]\mathcal{F}_q\{g\}(\boldsymbol{\omega})e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} [\mathcal{F}_q\{g\}(\boldsymbol{\omega})f_0(\boldsymbol{y}) + \mathbf{i}\mathcal{F}_q\{g\}(\boldsymbol{\omega})f_1(\boldsymbol{y}) \\
&\quad + \mathbf{j}\mathcal{F}_q\{g\}(\boldsymbol{\omega})f_2(\boldsymbol{y}) + \mathbf{k}\mathcal{F}_q\{g\}(\boldsymbol{\omega})f_3(\boldsymbol{y})]e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\
&= \mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}) \\
&\quad + \mathbf{j}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega}),
\end{aligned}$$

which completes the proof. □

Remark 3.1. *The quaternion convolution* (23) *can be expressed as*

$$f \star g = \frac{1}{(2\pi)^2}[g \star f_0 + \mathbf{i}(g \star f_1) + \mathbf{j}(g \star f_2) + \mathbf{k}(g \star f_3)]. \tag{26}$$

Corollary 1. *Let $f,\ g \in L^1(\mathbb{R}^2;\mathbb{H})$. Define*

$$h(\boldsymbol{x}) = \int_{\mathbb{R}^2} g(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}. \tag{27}$$

Then, $h \in L^1(\mathbb{R}^2;\mathbb{H})$ and

$$\begin{aligned}
(f \star h)(\boldsymbol{x}) &= \int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{i}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&\quad + \mathbf{j}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{k}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}.
\end{aligned} \tag{28}$$

Proof. A direct computation yields

$$\begin{aligned}
(f \star h)(\boldsymbol{x}) &= \int_{\mathbb{R}^2} f(\boldsymbol{y})h(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, d\boldsymbol{\omega} \right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [f_0(\boldsymbol{y}) + \mathbf{i}f_1(\boldsymbol{y}) + \mathbf{j}f_2(\boldsymbol{y}) + \mathbf{k}f_3(\boldsymbol{y})]g(\boldsymbol{\omega})e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [g(\boldsymbol{\omega})f_0(\boldsymbol{y}) + \mathbf{i}g(\boldsymbol{\omega})f_1(\boldsymbol{y})]e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y}\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\mathbf{j}g(\boldsymbol{\omega})f_2(\boldsymbol{y}) + \mathbf{k}g(\boldsymbol{\omega})f_3(\boldsymbol{y})]e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y}\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{i}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&\quad + \mathbf{j}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{k}\int_{\mathbb{R}^2} g(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega},
\end{aligned}$$

which completes the proof. □

We demonstrate some important properties of the convolution theorem of the type II QFT. We first establish the relationship between the type II QFT and conjugation of convolution of two quaternion-valued functions.

Theorem 3.2. *Let $f,\ g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then, we have*

$$\begin{aligned}\mathcal{F}_q\{\overline{f\star g}\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{g_0\}(\boldsymbol{\omega})\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega}) - \mathbf{i}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_1\}(\boldsymbol{\omega})\\ &\quad - \mathbf{j}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_2\}(\boldsymbol{\omega}) - \mathbf{k}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_3\}(\boldsymbol{\omega}).\end{aligned} \tag{29}$$

Proof. Applying the quaternion property and definition of the type II QFT, we have

$$\mathcal{F}_q\{\overline{f\star g}\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2}\bar{g}(\boldsymbol{y})\left[\int_{\mathbb{R}^2}\bar{f}(\boldsymbol{x}-\boldsymbol{y})e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d\boldsymbol{x}\right]d\boldsymbol{y}. \tag{30}$$

Put $\boldsymbol{z} = \boldsymbol{x}-\boldsymbol{y}$. Then, we have

$$\begin{aligned}\mathcal{F}_q\{\overline{f\star g}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}\bar{g}(\boldsymbol{y})\left[\int_{\mathbb{R}^2}\bar{f}(\boldsymbol{z})\,e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\,d\boldsymbol{z}\right]e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\,d\boldsymbol{y}\\ &= \int_{\mathbb{R}^2}\bar{g}(\boldsymbol{y})\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\,e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\,d\boldsymbol{y}.\end{aligned}$$

To overcome the noncommutativity of the quaternion multiplication, we decompose the quaternion function $\bar{g}$ into $g_0 - \mathbf{i}g_1 - \mathbf{j}g_2 - \mathbf{k}g_3$. This gives

$$\begin{aligned}\mathcal{F}_q\{\overline{f\star g}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2}[g_0(\boldsymbol{y}) - \mathbf{i}g_1(\boldsymbol{y}) - \mathbf{j}g_2(\boldsymbol{y}) - \mathbf{k}g_3(\boldsymbol{y})]\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\,e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\,d\boldsymbol{y}\\ &= \mathcal{F}_q\{g_0\}(\boldsymbol{\omega})\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega}) - \mathbf{i}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) - \mathbf{j}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_2\}(\boldsymbol{\omega})\\ &\quad - \mathbf{k}\mathcal{F}_q\{\bar{f}\}(\boldsymbol{\omega})\mathcal{F}_q\{g_3\}(\boldsymbol{\omega}),\end{aligned}$$

which completes the proof. □

The following lemma describes how the type II QFT behaves under the shift of quaternion convolution.

Lemma 1. *Let $f,\ g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then we have*

$$\begin{aligned}&\mathcal{F}_q\{\tau_{\boldsymbol{a}}(f\star g)\}(\boldsymbol{\omega})\\ &\quad= \mathcal{F}_q\{g\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{a}}\\ &\qquad + \mathbf{j}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{a}} + \mathbf{k}\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{a}}.\end{aligned} \tag{31}$$

The following lemma is important to solve partial differential equations in quaternion algebra.

Lemma 2. *Let $f, g \in L^1(\mathbb{R}^2;\mathbb{H})$. If $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2;\mathbb{R})$, then*

$$\mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}(\boldsymbol{\omega})\mathcal{F}_q\{g\}(\boldsymbol{\omega})\right](\boldsymbol{x}) = (g\star f)(\boldsymbol{x}), \tag{32}$$

or, equivalently,

$$\mathcal{F}_q\{g\star f\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})\mathcal{F}_q\{g\}(\boldsymbol{\omega}).$$

Proof. Since $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2;\mathbb{R})$, the type II QFT inversion implies

$$\begin{aligned}
\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}(\boldsymbol{\omega})\mathcal{F}_q\{g\}(\boldsymbol{\omega})](\boldsymbol{x}) &\overset{(22)}{=} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega})\, g(\boldsymbol{y})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y}\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2} g(\boldsymbol{y}) \left[\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, d\boldsymbol{\omega}\right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} g(\boldsymbol{y})\, f(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \\
&= (g \star f)(\boldsymbol{x}).
\end{aligned} \tag{33}$$

Here, in the second equality of (33), we can interchange the position because $\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2;\mathbb{R})$. This completes the proof of (32). □

Lemma 3. *Suppose that $f,\, g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then, we have the identity*

$$\begin{aligned}
\int_{\mathbb{R}^2} f(\boldsymbol{\omega})\, \mathcal{F}_q\{g\}(\boldsymbol{\omega})\, d\boldsymbol{\omega} &= \int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_0\}(\boldsymbol{y})\, d\boldsymbol{y} + \mathbf{i}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_1\}(\boldsymbol{y})\, d\boldsymbol{y} \\
&\quad + \mathbf{j}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_2\}(\boldsymbol{y})\, d\boldsymbol{y} + \mathbf{k}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_3\}(\boldsymbol{y})\, d\boldsymbol{y}.
\end{aligned} \tag{34}$$

Proof. By Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(\boldsymbol{\omega})\, \mathcal{F}_q\{g\}(\boldsymbol{\omega}) d\boldsymbol{\omega} \\
&\quad = \int_{\mathbb{R}^2} f(\boldsymbol{\omega}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} d\boldsymbol{y}\right] d\boldsymbol{\omega} \\
&\quad = \int_{\mathbb{R}^2} [f_0(\boldsymbol{\omega}) + \mathbf{i} f_1(\boldsymbol{\omega}) + \mathbf{j} f_2(\boldsymbol{\omega}) + \mathbf{k} f_3(\boldsymbol{\omega})] \left[\int_{\mathbb{R}^2} g(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} d\boldsymbol{y}\right] d\boldsymbol{\omega} \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [g(\boldsymbol{y}) f_0(\boldsymbol{\omega}) + \mathbf{i} g(\boldsymbol{y}) f_1(\boldsymbol{\omega}) + \mathbf{j} g(\boldsymbol{y}) f_2(\boldsymbol{\omega}) + \mathbf{k} g(\boldsymbol{y}) f_3(\boldsymbol{\omega})] e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}} d\boldsymbol{\omega}\, d\boldsymbol{y} \\
&\quad = \int_{\mathbb{R}^2} g(\boldsymbol{y}) \int_{\mathbb{R}^2} f_0(\boldsymbol{\omega}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} + \mathbf{i}\int_{\mathbb{R}^2} g(\boldsymbol{y}) f_1(\boldsymbol{\omega})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} \\
&\qquad + \mathbf{j}\int_{\mathbb{R}^2} g(\boldsymbol{y}) \int_{\mathbb{R}^2} f_2(\boldsymbol{\omega})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} + \mathbf{k}\int_{\mathbb{R}^2} g(\boldsymbol{y}) f_3(\boldsymbol{\omega})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} \\
&\quad = \int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_0\}(\boldsymbol{y})\, d\boldsymbol{y} + \mathbf{i}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_1\}(\boldsymbol{y})\, d\boldsymbol{y} \\
&\qquad + \mathbf{j}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_2\}(\boldsymbol{y})\, d\boldsymbol{y} + \mathbf{k}\int_{\mathbb{R}^2} g(\boldsymbol{y})\, \mathcal{F}_q\{f_3\}(\boldsymbol{y})\, d\boldsymbol{y},
\end{aligned} \tag{35}$$

which completes the proof. □

Note that Lemma 3 is a special case of the convolution theorem (24) with respect to a kernel. Because the kenel can be written as an L^2 quaternion Fourier integral. More precisely,

$$K(\boldsymbol{x}) = \int_{\mathbb{R}^2} k(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}$$

implies

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(\boldsymbol{x}-\boldsymbol{y}) K(\boldsymbol{y})\, d\boldsymbol{y} \\
&\quad = \int_{\mathbb{R}^2} f(\boldsymbol{x}-\boldsymbol{y}) \int_{\mathbb{R}^2} k(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} f(\boldsymbol{x}-\boldsymbol{y}) \int_{\mathbb{R}^2} k(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{\omega}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_0(\boldsymbol{z}) e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{z})}\, d\boldsymbol{z}\, d\boldsymbol{\omega} + \mathbf{i} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) f_1(\boldsymbol{z})\, e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{z})}\, d\boldsymbol{z}\, d\boldsymbol{\omega} \\
&\quad + \mathbf{j} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_2(\boldsymbol{z})\, e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{z})}\, d\boldsymbol{z}\, d\boldsymbol{\omega} + \mathbf{k} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_3(\boldsymbol{z})\, e^{\mu\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{z})}\, d\boldsymbol{z}\, d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \mathcal{F}_q\{f_0\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{i} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&\quad + \mathbf{j} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \mathcal{F}_q\{f_2\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} + \mathbf{k} \int_{\mathbb{R}^2} k(\boldsymbol{\omega}) \mathcal{F}_q\{f_3\}(\boldsymbol{\omega})\, e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega}. \qquad (36)
\end{aligned}
$$

Theorem 3.3. *Let $f,\, g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then, we have*

$$
\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \mathcal{F}_q\{g_{c^0(\mu)}\}(\boldsymbol{\omega}) + \mathcal{F}_q\{f\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g_{c^1(\mu)}\}(\boldsymbol{\omega}). \qquad (37)
$$

Proof. Indeed, we have

$$
\begin{aligned}
\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{x}-\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x}\right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} (g_{c^0(\mu)}(\boldsymbol{z}) + g_{c^1(\mu)}(\boldsymbol{z}))\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z}\right] e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} g_{c^0(\mu)}(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \\
&\qquad + \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} g_{c^1(\mu)}(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \\
&= \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \mathcal{F}_q\{g_{c^0(\mu)}\}(\boldsymbol{\omega}) + \mathcal{F}_q\{f\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g_{c^1(\mu)}\}(\boldsymbol{\omega}), \qquad (38)
\end{aligned}
$$

which finishes the proof. □

An alternative form of Theorem 3.2 is given by

Theorem 3.4. *Let $f,\, g \in L^1(\mathbb{R}^2;\mathbb{H})$. Then, we have*

$$
\begin{aligned}
\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \mathcal{F}_q\{g\}(\boldsymbol{\omega}) - \mathcal{F}_q\{f\mu\}(\boldsymbol{\omega}) \mathcal{F}_q\{g\mu\}(\boldsymbol{\omega}) \\
&\quad + \mathcal{F}_q\{f\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g\}(\boldsymbol{\omega}) + \mathcal{F}_q\{f\mu\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g\mu\}(\boldsymbol{\omega}). \qquad (39)
\end{aligned}
$$

Proof. A simple computation shows that

$$
\begin{aligned}
&\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} g(\boldsymbol{x}-\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x}\right] d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) \left[\int_{\mathbb{R}^2} (g_{c^0(\mu)}(\boldsymbol{z}) + g_{c^1(\mu)}(\boldsymbol{z}))\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z}\right] e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} g_{c^0(\mu)}(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \\
&\qquad + \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} g_{c^1(\mu)}(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} (g(\boldsymbol{z}) - \mu g(\boldsymbol{z})\mu)\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \\
&\qquad + \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \int_{\mathbb{R}^2} (g(\boldsymbol{z}) + \mu g(\boldsymbol{z})\mu)\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \left(\int_{\mathbb{R}^2} g(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} - \mu \int_{\mathbb{R}^2} g(\boldsymbol{z}) \mu\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \right) \\
&\qquad + \int_{\mathbb{R}^2} f(\boldsymbol{y}) e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \left(\int_{\mathbb{R}^2} g(\boldsymbol{z})\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} + \mu \int_{\mathbb{R}^2} g(\boldsymbol{z}) \mu\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{z}}\, d\boldsymbol{z} \right) \\
&= \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \mathcal{F}_q\{g\}(\boldsymbol{\omega}) - \mathcal{F}_q\{f\mu\}(\boldsymbol{\omega}) \mathcal{F}_q\{g\mu\}(\boldsymbol{\omega}) \\
&\qquad + \mathcal{F}_q\{f\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g\}(\boldsymbol{\omega}) + \mathcal{F}_q\{f\mu\}(-\boldsymbol{\omega}) \mathcal{F}_q\{g\mu\}(\boldsymbol{\omega}),
\end{aligned} \tag{40}
$$

which finishes the proof. □

Remark 3.2. *Apply the inverse quaternion Fourier transform to the both sides of* (39), *we have*

$$
f \star g = \frac{1}{(2\pi)^2} [f \star g - f\mu \star g\mu + f\mu \star g\mu + f\mu \star g\mu]. \tag{41}
$$

The following theorem provides a general form of Theorem 3.7 in [12].

Theorem 3.5. *Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$. Then we have*

$$
\mathcal{F}_q\{fg\}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^2} ((\mathcal{F}_q\{f\} * \mathcal{F}_q\{g_{c^0\mu}\})(\boldsymbol{\omega}) + (\overline{\mathcal{F}_q\{f\}} \circ \mathcal{F}_q\{g_{c^1\mu}\})(\boldsymbol{\omega})). \tag{42}
$$

Proof. Straightforward computation gives

$$
\begin{aligned}
&\mathcal{F}_q\{fg\}(\boldsymbol{\omega}) \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{x}) g(\boldsymbol{x}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \left(\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{v}) e^{\mu\boldsymbol{v}\cdot\boldsymbol{x}}\, d\boldsymbol{u} \right) g(\boldsymbol{x}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \left(\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{v}) e^{\mu\boldsymbol{v}\cdot\boldsymbol{x}}\, d\boldsymbol{u} \right) (g_{c^0\mu}(\boldsymbol{x}) + g_{c^1\mu}(\boldsymbol{x})) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) g_{c^0\mu}(\boldsymbol{x})\, e^{\mu\boldsymbol{v}\cdot\boldsymbol{x}} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{v}\, d\boldsymbol{x} \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) g_{c^1\mu}(\boldsymbol{x})\, e^{\mu\boldsymbol{v}\cdot\boldsymbol{x}} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{v}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^4} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^0\mu}\}(\boldsymbol{v})\, e^{\mu\boldsymbol{u}\cdot\boldsymbol{x}} e^{\mu\boldsymbol{v}\cdot\boldsymbol{x}} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{u}\, d\boldsymbol{v}\, d\boldsymbol{x} \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^4} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^1\mu}\}(\boldsymbol{v})\, e^{\mu\boldsymbol{u}\cdot\boldsymbol{x}} e^{-\mu\boldsymbol{v}\cdot\boldsymbol{x}} e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{u}\, d\boldsymbol{v}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^0\mu}\}(\boldsymbol{v}) \delta(\boldsymbol{v} + \boldsymbol{u} - \boldsymbol{\omega})\, d\boldsymbol{v}\, d\boldsymbol{u} \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^1\mu}\}(\boldsymbol{v}) \delta(\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{\omega})\, d\boldsymbol{v}\, d\boldsymbol{u} \\
&= \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^0\mu}\}(\boldsymbol{\omega} - \boldsymbol{u})\, d\boldsymbol{u} \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q\{g\}(\boldsymbol{u}) \mathcal{F}_q\{g_{c^1\mu}\}(\boldsymbol{\omega} + \boldsymbol{u})\, d\boldsymbol{u},
\end{aligned} \tag{43}
$$

which completes the proof. □

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